Universal character and q-difference Painlevé equations with affine Weyl groups

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Abstract

The universal character is a polynomial attached to a pair of partitions and is a generalization of the Schur polynomial. In this paper, we introduce an integrable system of q-difference lattice equations satisfied by the universal character, and call it the *lattice q-UC hierarchy*. We regard it as generalizing both q-KP and q-UC hierarchies. Suitable similarity and periodic reductions of the hierarchy yield the q-difference Painlevé equations of types $A^{(1)}_{2g+1}$ ($g \geq 1$), $D^{(1)}_5$, and $E^{(1)}_6$. As its consequence, a class of algebraic solutions of the q-Painlevé equations is rapidly obtained by means of the universal character. In particular, we demonstrate explicitly the reduction procedure for the case of type $E^{(1)}_6$, via the framework of τ-functions based on the geometry of certain rational surfaces.

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1 Introduction

The present article is aimed to clarify the underlying relationship between the universal character and the q-difference Painlevé equations from the viewpoint of infinite integrable systems.

The universal character $S_{[\lambda,\mu]}$, defined by K. Koike [6], is a polynomial attached to a pair of partitions $[\lambda, \mu]$ and is a generalization of the Schur polynomial S_λ. The universal character describes the character of an irreducible rational representation of the general linear group, while the Schur polynomial, as is well-known, does that of an irreducible polynomial representation; see [6], for details.

The algebraic theory of the KP hierarchy of nonlinear partial-differential equations is probably the most beautiful one in the field of classical integrable systems. It was discovered by M. Sato that the totality of solutions of the KP hierarchy forms an infinite-dimensional Grassmann manifold; in particular, the set of homogeneous polynomial solutions coincides with the whole set of Schur polynomials; see [9, 14]. We say that the KP hierarchy is an infinite integrable system which characterizes the Schur polynomials. On the other hand, an extension of the KP hierarchy called the *UC hierarchy* was proposed by the author [15]; it is an infinite integrable system characterizing the universal characters as its homogeneous polynomial solutions (see the table below).

Character polynomials	versus	Infinite integrable systems
Schur polynomial S_λ		KP hierarchy
$\cap$		$\cap$
Universal character $S_{[\lambda,\mu]}$		UC hierarchy

In this paper, we first introduce an integrable system of q-difference equations defined on two-dimensional lattice, called the *lattice q-UC hierarchy* (see Definition 2.2). It is considered as generalizing both q-KP and q-UC hierarchies, which are the q-analogues of the KP and UC hierarchies; *cf.* [5] and [17] (see Remark 2.5). Next we show that suitable similarity and periodic reductions of the lattice q-UC hierarchy yield the q-Painlevé equations with affine Weyl group symmetries. Let us refer each of q-Painlevé equations by the Dynkin diagram of associated root system; for example, the q-Painlevé VI equation is represented by $D_5^{(1)}$; see [1, 13]. Then our main result is stated as follows:

Theorem 1.1. *The q-Painlevé equations of types $A_{2g+1}^{(1)}$ $(g \geq 1)$, $D_5^{(1)}$, and $E_6^{(1)}$ can be obtained as certain similarity reductions of the lattice q-UC hierarchy with the periodic conditions of order* $(g+1, g+1)$, $(2, 2)$, *and* $(3, 3)$, *respectively.*

We shall demonstrate the proof of the above theorem in detail, particularly for the case of type $E_6^{(1)}$; the other cases are briefly studied in Appendix.

Recall that the (higher order) q-Painlevé equation of type $A_{N-1}^{(1)}$ is a further generalization of q-Painlevé IV and V equations which correspond to the cases $N = 3$ and 4, respectively; see [4, 8]. As shown in [5], it can also be obtained as a similarity reduction of the q-KP hierarchy with N-periodicity. With this fact in mind, we summarize in the following table how the q-Painlevé equations relate to the similarity reductions of q-KP or lattice q-UC hierarchies with periodic conditions:

q-Painlevé equation	$A_{2g}^{(1)}$	$A_{2g+1}^{(1)}$	$D_5^{(1)}$	$E_6^{(1)}$
q-KP hierarchy	$2g+1$	$2g+2$	–	–
Lattice q-UC hierarchy	–	$(g+1, g+1)$	$(2, 2)$	$(3, 3)$

The universal characters are homogeneous solutions of the lattice q-UC hierarchy (see Proposition 2.3). Hence we have immediately from Theorem 1.1 a class of algebraic solutions of the q-Painlevé equations in terms of the universal character.

Corollary 1.2. *The q-Painlevé equations of types $A^{(1)}_{2g+1}$ $(g \geq 1)$, $D^{(1)}_5$, and $E^{(1)}_6$ admit a class of algebraic solutions expressed in terms of the universal characters attached to pairs of* $(g+1)$-, 2-, *and* 3-*core partitions, respectively.*

Remark 1.3. (i) In K. Kajiwara *et al.* [5], rational solutions of the q-Painlevé equations of type $A^{(1)}_{N-1}$ were constructed by means of the Schur polynomial attached to an N-core partition, via the similarity reduction of the q-KP hierarchy.
(ii) We investigated certain similarity reductions of the q-UC hierarchy and already obtained the same class of solutions as above for the cases $A^{(1)}_{2g+1}$ and $D^{(1)}_5$; see [17] and [18]. Also, for $A^{(1)}_3$ (the q-Painlevé V equation), the rational solutions were firstly found by T. Masuda [8] without concerning any relationship to the infinite integrable systems.
(iii) It is still an interesting open problem to obtain the q-Painlevé equations of types $E^{(1)}_7$ and $E^{(1)}_8$ as reductions of some integrable hierarchies such as KP, UC, or beyond.

In Section 2, we introduce the lattice q-UC hierarchy, which is an integrable system of q-difference lattice equations satisfied by the universal characters (Definition 2.2 and Proposition 2.3). In Section 3, we present a birational representation of affine Weyl group of type $E^{(1)}_6$ defined over the field of τ-*functions*, starting from a certain configuration of nine points in the complex projective plane (Theorem 3.2). Then we define the q-Painlevé equation of type $E^{(1)}_6$ (q-$P(E_6)$) by means of the translation part of the affine Weyl group (Definition 3.3). Section 4 concerns the system of bilinear equations satisfied by τ-functions (Proposition 4.2). In Section 5, we show that the bilinear form of q-$P(E_6)$ coincides with a similarity reduction of the lattice q-UC hierarchy. Consequently, in Section 6, we have a class of algebraic solutions of q-$P(E_6)$ in terms of the universal character (Theorem 6.2). Section 7 is devoted to the proof of Proposition 2.3. We briefly sum up in Appendix results on the reductions to the q-Painlevé equations of types $A^{(1)}_{2g+1}$ and $D^{(1)}_5$.

Note. Throughout this paper, we shall use the following convention of *q-shifted factorials*:

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), \quad (a;p,q)_\infty = \prod_{i,j=0}^{\infty}(1-ap^iq^j),$$

and also $(a_1,\dots,a_r;q)_\infty = (a_1;q)_\infty \cdots (a_r;q)_\infty$.

2 Universal characters and lattice q-UC hierarchy

2.1 Universal characters

For a pair of sequences of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{l'})$, the *universal character* $S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y})$ is a polynomial in $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$ defined by the determinant formula of *twisted* Jacobi–Trudi type (see [6]):

$$S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = \det \begin{pmatrix} p_{\mu_{l'-i+1}+i-j}(\boldsymbol{y}), & 1 \le i \le l' \\ p_{\lambda_{i-l'}-i+j}(\boldsymbol{x}), & l'+1 \le i \le l+l' \end{pmatrix}_{1 \le i,j \le l+l'}, \tag{2.1}$$

where p_n is a polynomial defined by the generating function:

$$\sum_{k \in \mathbb{Z}} p_k(\boldsymbol{x}) z^k = \exp\left(\sum_{n=1}^{\infty} x_n z^n \right). \tag{2.2}$$

Schur polynomial $S_\lambda(\boldsymbol{x})$ (see [7]) is regarded as a special case of the universal character:

$$S_\lambda(\boldsymbol{x}) = \det(p_{\lambda_i - i + j}(\boldsymbol{x})) = S_{[\lambda,\emptyset]}(\boldsymbol{x}, \boldsymbol{y}).$$

If we count the degree of variables as $\deg x_n = n$ and $\deg y_n = -n$, then the universal character $S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y})$ is a weighted homogeneous polynomial of degree $|\lambda| - |\mu|$, where $|\lambda| = \lambda_1 + \cdots + \lambda_l$. Namely, we have

$$S_{[\lambda,\mu]}(cx_1, c^2 x_2, \ldots, c^{-1} y_1, c^{-2} y_2, \ldots) = c^{|\lambda|-|\mu|} S_{[\lambda,\mu]}(x_1, x_2, \ldots, y_1, y_2, \ldots), \tag{2.3}$$

for any nonzero constant c.

Example 2.1. When $\lambda = (2, 1)$ and $\mu = (1)$, the universal character is given by

$$S_{[(2,1),(1)]}(\boldsymbol{x}, \boldsymbol{y}) = \begin{vmatrix} p_1(\boldsymbol{y}) & p_0(\boldsymbol{y}) & p_{-1}(\boldsymbol{y}) \\ p_1(\boldsymbol{x}) & p_2(\boldsymbol{x}) & p_3(\boldsymbol{x}) \\ p_{-1}(\boldsymbol{x}) & p_0(\boldsymbol{x}) & p_1(\boldsymbol{x}) \end{vmatrix} = \left(\frac{{x_1}^3}{3} - x_3 \right) y_1 - {x_1}^2.$$

2.2 Lattice q-UC hierarchy

Let $I \subset \mathbb{Z}_{>0}$ and $J \subset \mathbb{Z}_{<0}$ be finite indexing sets and t_i $(i \in I \cup J)$ the independent variables. Let $T_i = T_{i;q}$ be the q-shift operator defined by

$$T_{i;q}(t_i) = \begin{cases} qt_i & (i \in I), \\ q^{-1} t_i & (i \in J), \end{cases}$$

and $T_{i;q}(t_j) = t_j$ $(i \neq j)$. We use also the notation: $T_{i_1} T_{i_2} \cdots T_{i_n} = T_{i_1 i_2 \ldots i_n}$, for the sake of brevity.

Definition 2.2. The following system of q-difference equations for unknowns $\sigma_{m,n}(\boldsymbol{t})$ $(m, n \in \mathbb{Z})$ is called the *lattice q-UC hierarchy*:

$$t_i T_i(\sigma_{m,n+1}) T_j(\sigma_{m+1,n}) - t_j T_j(\sigma_{m,n+1}) T_i(\sigma_{m+1,n}) = (t_i - t_j) T_{ij}(\sigma_{m,n}) \sigma_{m+1,n+1}, \tag{2.4}$$

where $i, j \in I \cup J$.

Let us consider the change of variables

$$x_n = \frac{\sum_{i\in I} t_i{}^n - q^n \sum_{j\in J} t_j{}^n}{n(1-q^n)}, \quad y_n = \frac{\sum_{i\in I} t_i{}^{-n} - q^{-n} \sum_{j\in J} t_j{}^{-n}}{n(1-q^{-n})}, \tag{2.5}$$

then define the symmetric function $s_{[\lambda,\mu]} = s_{[\lambda,\mu]}(\boldsymbol{t})$ in t_i ($i \in I \cup J$) by

$$s_{[\lambda,\mu]}(\boldsymbol{t}) = S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}). \tag{2.6}$$

The universal characters solve the lattice q-UC hierarchy in the following sense.

Proposition 2.3. *We have*

$$\begin{aligned} &t_i T_i(s_{[\lambda,(k',\mu)]}) T_j(s_{[(k,\lambda),\mu]}) - t_j T_j(s_{[\lambda,(k',\mu)]}) T_i(s_{[(k,\lambda),\mu]}) \\ &= (t_i - t_j) T_{ij}(s_{[\lambda,\mu]}) s_{[(k,\lambda),(k',\mu)]}, \end{aligned} \tag{2.7}$$

for any integers k, k' and sequences of integers $\lambda = (\lambda_1, \ldots, \lambda_l), \mu = (\mu_1, \ldots, \mu_{l'})$.

The proof of the proposition above will be given in Section 7.

Remark 2.4. Define the functions $h_n = h_n(\boldsymbol{t})$ and $H_n = H_n(\boldsymbol{t})$ by

$$h_n(\boldsymbol{t}) = p_n(\boldsymbol{x}), \quad H_n(\boldsymbol{t}) = p_n(\boldsymbol{y}),$$

under (2.5). We note also the following expression by the generating functions:

$$\sum_{k=0}^{\infty} h_k(\boldsymbol{t}) z^k = \prod_{i\in I, j\in J} \frac{(q t_j z; q)_\infty}{(t_i z; q)_\infty}, \quad \sum_{k=0}^{\infty} H_k(\boldsymbol{t}) z^k = \prod_{i\in I, j\in J} \frac{(q^{-1} t_j{}^{-1} z; q^{-1})_\infty}{(t_i{}^{-1} z; q^{-1})_\infty}. \tag{2.8}$$

Hence, function $s_{[\lambda,\mu]}(\boldsymbol{t})$ can be expressed as

$$s_{[\lambda,\mu]}(\boldsymbol{t}) = \det \begin{pmatrix} H_{\mu_{l'-i+1}+i-j}(\boldsymbol{t}), & 1 \le i \le l' \\ h_{\lambda_{i-l'}-i+j}(\boldsymbol{t}), & l'+1 \le i \le l+l' \end{pmatrix}_{1\le i,j\le l+l'}. \tag{2.9}$$

Remark 2.5. (i) One can easily deduce from (2.4) the following equation:

$$\begin{aligned} &(t_i - t_j) T_{ij}(\sigma_{m,n}) T_k(\sigma_{m+1,n}) + (t_j - t_k) T_{jk}(\sigma_{m,n}) T_i(\sigma_{m+1,n}) \\ &+ (t_k - t_i) T_{ik}(\sigma_{m,n}) T_j(\sigma_{m+1,n}) = 0, \end{aligned} \tag{2.10}$$

where $i, j, k \in I \cup J$, which is exactly the bilinear equation of the q-UC hierarchy; see [17].
(ii) If $\sigma_{m,n}(\boldsymbol{t})$ does not depend on n, that is, $\sigma_{m,n} = \sigma_{m,n+1}$ for all m and n, then (2.4) is reduced to the q-KP hierarchy (see [5]):

$$t_i T_i(\rho_m) T_j(\rho_{m+1}) - t_j T_j(\rho_m) T_i(\rho_{m+1}) = (t_i - t_j) T_{ij}(\rho_m) \rho_{m+1}, \tag{2.11}$$

where $\rho_m := \sigma_{m,n}$.

3 τ-functions of q-Painlevé equation

In this section we present a geometric formulation of the q-Painlevé equation of type $E_6^{(1)}$ by means of τ-functions; *cf.* [13]. Consider the configuration of nine points in the complex projective plane $\mathbb{P}^2$, which are divided into three triples of collinear points. Let $[x : y : z]$ be the homogeneous coordinate of $\mathbb{P}^2$. We can normalize, without loss of generality, the nine points p_i $(1 \le i \le 9)$ under consideration as follows:

$$\begin{array}{lll} p_1 = [0 : -1 : a_3], & p_2 = [0 : -1 : a_3{a_6}^3], & p_3 = [0 : -1 : a_3{a_6}^3{a_0}^3], \\ p_4 = [a_3 : 0 : -1], & p_5 = [{a_2}^3 a_3 : 0 : -1], & p_6 = [{a_1}^3{a_2}^3 a_3 : 0 : -1], \\ p_7 = [-1 : a_3 : 0], & p_8 = [-1 : a_3{a_4}^3 : 0], & p_9 = [-1 : a_3{a_4}^3{a_5}^3 : 0], \end{array} \tag{3.1}$$

where $a_i \in \mathbb{C}^\times$ are parameters such that $a_0 a_1 {a_2}^2 {a_3}^3 {a_4}^2 a_5 {a_6}^2 = q$. Let $\psi : X = X_{\boldsymbol{a}} \to \mathbb{P}^2$ be the blowing-up at the nine points. Let $e_i = \psi^{-1}(p_i)$ be the exceptional divisor and h the divisor class corresponding to a hyperplane. We thus have the Picard lattice:

$$\mathrm{Pic}(X) = \mathbb{Z}h \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_9,$$

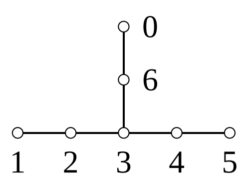
of rational surface X, equipped with the intersection form $(\ |\ )$ defined by $(h|h) = 1$, $(e_i|e_j) = -\delta_{i,j}$ and $(h|e_j) = 0$. The anti-canonical divisor $-K_X$ is uniquely decomposed into prime divisors:

$$-K_X = 3h - \sum_{1 \le i \le 9} e_i = D_1 + D_2 + D_3,$$

where $D_1 = h - e_1 - e_2 - e_3$, $D_2 = h - e_4 - e_5 - e_6$ and $D_3 = h - e_7 - e_8 - e_9$. Since the dual graph of the intersections of D_i's is of type $A_2^{(1)}$, we call X the $A_2^{(1)}$-*surface* following the classification of the *generalized Halphen surfaces* due to H. Sakai [13]. The orthogonal complement $(-K_X)^\perp \overset{\text{def}}{=} \{v \in \mathrm{Pic}(X) \,|\, (v|D_i) = 0 \text{ for } i = 1, 2, 3\}$ is isomorphic to the root lattice of type $E_6^{(1)}$. In fact, $(-K_X)^\perp$ is generated by the vectors $\alpha_{ij} = e_i - e_j$ (where both i and j belong to the same indexing set $\{1, 2, 3\}$, $\{4, 5, 6\}$, or $\{7, 8, 9\}$) and $\alpha_{ijk} = h - e_i - e_j - e_k$ $(i \le 3 < j \le 6 < k)$; hence we can choose a root basis $B = \{\alpha_0, \ldots, \alpha_6\}$ defined by

$$\alpha_0 = \alpha_{23}, \quad \alpha_1 = \alpha_{56}, \quad \alpha_2 = \alpha_{45}, \quad \alpha_3 = \alpha_{147}, \quad \alpha_4 = \alpha_{78}, \quad \alpha_5 = \alpha_{89}, \quad \alpha_6 = \alpha_{12},$$

whose Dynkin diagram is of type $E_6^{(1)}$ and looks as follows (see, *e.g.*, [2]):



Note that the 72 roots of E_6 are represented by α_{ij} (18 vectors) and $\pm\alpha_{ijk}$ (54 vectors). We define the action of the reflection corresponding to a root $\alpha \in (-K_X)^\perp$ by

$$r_\alpha(v) = v + (v|\alpha)\alpha, \quad v \in \mathrm{Pic}(X).$$

We prepare the notations, $r_{ij} := r_{\alpha_{ij}}$, $r_{ijk} := r_{\alpha_{ijk}}$ and $s_i := r_{\alpha_i}$ ($i = 0, \dots, 6$), for convenience. Also, the diagram automorphism ι_i ($i = 1, 2$) is defined by

$$\iota_1(e_{\{1,2,3,7,8,9\}}) = e_{\{7,8,9,1,2,3,\}}, \quad \iota_2(e_{\{1,2,3,4,5,6\}}) = e_{\{4,5,6,1,2,3,\}}.$$

We thus obtain the linear action of the (extended) affine Weyl group $\widetilde{W}(E_6^{(1)}) = \langle s_0, \dots, s_6, \iota_1, \iota_2 \rangle$ on $\mathrm{Pic}(X)$. In parallel, we fix the action of $\widetilde{W}(E_6^{(1)})$ on the *multiplicative* root variables $\boldsymbol{a} = (a_0, \dots, a_6)$ as follows:

$$\begin{aligned} &s_i(a_j) = a_j a_i^{-C_{ij}}, \\ &\iota_1(a_{\{0,1,2,3,4,5,6\}}) = a_{\{5,1,2,3,6,0,4\}}{}^{-1}, \quad \iota_2(a_{\{0,1,2,3,4,5,6\}}) = a_{\{1,0,6,3,4,5,2\}}{}^{-1}, \end{aligned} \tag{3.2}$$

where (C_{ij}) being the Cartan matrix of type $E_6^{(1)}$.

Next we shall extend the linear action above to birational transformations. To this end, we introduce the notion of τ-functions; *cf.* [3]. Consider the field $\mathcal{L} = K(\tau_1, \dots, \tau_9)$ of rational functions in indeterminates τ_i ($1 \le i \le 9$) with the coefficient field $K = \mathbb{C}(\boldsymbol{a}^{1/3}) = \mathbb{C}(a_0{}^{1/3}, \dots, a_6{}^{1/3})$. Take a sub-lattice $M = \bigcup_{i=1,2,3} M_i$ of $\mathrm{Pic}(X)$, where

$$M_i = \left\{ v \in \mathrm{Pic}(X) \,\middle|\, (v|v) = -(v|D_i) = -1,\ (v|D_j) = 0\ (j \neq i) \right\}.$$

Definition 3.1. A function $\tau : M \to \mathcal{L}$ is said to be a *τ-function* iff it satisfies the conditions: (i) $\tau(w.v) = w.\tau(v)$ for any $v \in M$ and $w \in \widetilde{W}(E_6^{(1)})$; (ii) $\tau(e_i) = \tau_i$ ($1 \le i \le 9$).

Such functions and the action of $\widetilde{W}(E_6^{(1)})$ on them are explicitly determined in the following way. Any divisor $\Lambda = nh - e_{i_1} - \cdots - e_{i_{n+1}} \in M$ corresponds to a curve of degree n on $\mathbb{P}^2$ passing through $n+1$ points $p_{i_1}, \dots, p_{i_{n+1}}$ (with counting the multiplicity). We can choose uniquely the *normalized* defining polynomial $F_\Lambda(x, y, z) = \sum_{i+j+k=n} A_{ijk} x^i y^j z^k \in \mathbb{Q}(\boldsymbol{a})[x, y, z]$ of the curve, such that $\prod A_{ijk} = 1$. For example, we have

$$\begin{aligned} F_{h-e_1-e_4} &= a_3{}^{-1} x + a_3 y + z, \\ F_{h-e_4-e_7} &= x + a_3{}^{-1} y + a_3 z, \\ F_{h-e_1-e_7} &= a_3 x + y + a_3{}^{-1} z. \end{aligned}$$

Let

$$\left(\frac{x}{c_x}, \frac{y}{c_y}, \frac{z}{c_z} \right) = (\tau_1\tau_2\tau_3, \tau_4\tau_5\tau_6, \tau_7\tau_8\tau_9), \tag{3.3}$$

where

$$c_x = a_1{}^{\frac{1}{3}} a_2{}^{\frac{2}{3}} a_4{}^{-\frac{2}{3}} a_5{}^{-\frac{1}{3}}, \quad c_y = a_5{}^{\frac{1}{3}} a_4{}^{\frac{2}{3}} a_6{}^{-\frac{2}{3}} a_0{}^{-\frac{1}{3}}, \quad c_z = a_0{}^{\frac{1}{3}} a_6{}^{\frac{2}{3}} a_2{}^{-\frac{2}{3}} a_1{}^{-\frac{1}{3}}.$$

Suppose that

$$F_\Lambda(x, y, z) = \tau(nh - e_{i_1} - \cdots - e_{i_{n+1}}) \tau(e_{i_1}) \cdots \tau(e_{i_{n+1}}). \tag{3.4}$$

Therefore we see that the linear action of $\widetilde{W}(E_6^{(1)})$ on M yields the action on τ-functions immediately. For instance, by using $r_{ijk}(e_k) = h - e_i - e_j$, we can compute the action of r_{ijk}:

$$r_{ijk}(\tau(e_k)) = \tau(h - e_i - e_j) = \frac{F_{h-e_i-e_j}(x, y, z)}{\tau(e_i)\tau(e_j)}.$$

Each action of r_{ij} and diagram automorphism ι_i is realized as just a permutation of τ_i's. Summarizing above, we now arrive at the following theorem.

Theorem 3.2. *Define the birational transformations* s_i $(0 \le i \le 6)$ *and* ι_j $(j = 1, 2)$ *on* $\mathcal{L} = \mathbb{C}(\boldsymbol{a}^{1/3})(\tau_1, \dots, \tau_9)$ *by*

$$\begin{aligned}
&s_1(\tau_{\{5,6\}}) = \tau_{\{6,5\}}, \quad s_2(\tau_{\{4,5\}}) = \tau_{\{5,4\}}, \quad s_4(\tau_{\{7,8\}}) = \tau_{\{8,7\}}, \quad s_5(\tau_{\{8,9\}}) = \tau_{\{9,8\}},\\
&s_6(\tau_{\{1,2\}}) = \tau_{\{2,1\}}, \quad s_0(\tau_{\{2,3\}}) = \tau_{\{3,2\}}, \quad \iota_1(\tau_{\{1,2,3\}}) = \tau_{\{7,8,9\}}, \quad \iota_2(\tau_{\{1,2,3\}}) = \tau_{\{4,5,6\}},\\
&s_3(\tau_1) = \left(c_x\tau_1\tau_2\tau_3 + {a_3}^{-1}c_y\tau_4\tau_5\tau_6 + a_3c_z\tau_7\tau_8\tau_9\right)/(\tau_4\tau_7),\\
&s_3(\tau_4) = \left(a_3c_x\tau_1\tau_2\tau_3 + c_y\tau_4\tau_5\tau_6 + {a_3}^{-1}c_z\tau_7\tau_8\tau_9\right)/(\tau_1\tau_7),\\
&s_3(\tau_7) = \left({a_3}^{-1}c_x\tau_1\tau_2\tau_3 + a_3c_y\tau_4\tau_5\tau_6 + c_z\tau_7\tau_8\tau_9\right)/(\tau_1\tau_4).
\end{aligned} \tag{3.5}$$

Then (3.5) *with* (3.2) *provide a realization of* $\widetilde{W}(E_6^{(1)}) = \langle s_0, \dots, s_6, \iota_1, \iota_2 \rangle$.

Let

$$[f : g : 1] = \left[\frac{x}{c_x} : \frac{y}{c_y} : \frac{z}{c_z}\right] = [\tau_1\tau_2\tau_3 : \tau_4\tau_5\tau_6 : \tau_7\tau_8\tau_9]. \tag{3.6}$$

By virtue of Theorem 3.2, we obtain the following birational transformations on the inhomogeneous coordinate (f, g):

$$\begin{aligned}
&s_3(f) = f\frac{c_xf + {a_3}^{-1}c_yg + a_3c_z}{{a_3}^{-1}c_xf + a_3c_yg + c_z},\\
&s_3(g) = g\frac{a_3c_xf + c_yg + {a_3}^{-1}c_z}{{a_3}^{-1}c_xf + a_3c_yg + c_z},\\
&\iota_1(f) = \frac{1}{f}, \quad \iota_1(g) = \frac{g}{f}, \quad \iota_2(f) = g, \quad \iota_2(g) = f.
\end{aligned} \tag{3.7}$$

The birational action arising from the translation part of affine Weyl group can be regarded as a discrete dynamical system and is called a discrete Painlevé equation; *cf.* [12]. Consider an element

$$\ell = r_{258}r_{369}r_{258}r_{147} = (s_2s_4s_6s_0s_1s_5s_3s_2s_4s_6s_3)^2 \in W(E_6^{(1)}), \tag{3.8}$$

acting on the parameters $\boldsymbol{a} = (a_0, \dots, a_6)$ as their q-shifts:

$$\ell(\boldsymbol{a}) = \overline{\boldsymbol{a}} = (a_0, a_1, q^{-1}a_2, q^2a_3, q^{-1}a_4, a_5, q^{-1}a_6). \tag{3.9}$$

We define rational functions $F(\boldsymbol{a}; f, g),\ G(\boldsymbol{a}; f, g) \in \mathbb{C}(\boldsymbol{a}^{1/3}; f, g)$ by

$$\ell(f) = F(\boldsymbol{a}; f, g), \quad \ell(g) = G(\boldsymbol{a}; f, g). \tag{3.10}$$

Definition 3.3. The system of functional equations

$$f(\overline{\boldsymbol{a}}) = F(\boldsymbol{a}; f(\boldsymbol{a}), g(\boldsymbol{a})), \quad g(\overline{\boldsymbol{a}}) = G(\boldsymbol{a}; f(\boldsymbol{a}), g(\boldsymbol{a})), \tag{3.11}$$

for unknowns $f = f(\boldsymbol{a})$ and $g = g(\boldsymbol{a})$ is called the *q-Painlevé equation of type $E_6^{(1)}$*.

We shall often denote (3.11) shortly by q-$P(E_6)$.

Remark 3.4. We have the following inclusion relation of affine Weyl groups: $W(E_6^{(1)}) \supset W(A_5^{(1)}) \oplus W(A_1^{(1)})$. For instance, the sets of vectors $B' = \{\alpha_{158}, \alpha_{367}, \alpha_{248}, \alpha_{169}, \alpha_{257}, \alpha_{349}\}$ and $B'' = \{\alpha_{147}, \alpha_{258} + \alpha_{369}\}$ realize the root bases of types $A_5^{(1)}$ and $A_1^{(1)}$, respectively. Moreover, they are mutually orthogonal. The transformation ℓ, used to define the q-Painlevé equation (3.11), is exactly the translation in $W(A_1^{(1)})$; that is, $r_{\alpha_{258}+\alpha_{369}}r_{\alpha_{147}} = (r_{258}r_{369}r_{258})r_{147} = \ell$.

4 Bilinear equations among τ-functions

Let us introduce the transformations $\ell_2 = r_{369}r_{147}r_{369}r_{258}$ and $\ell_3 = r_{147}r_{258}r_{147}r_{369}$, in parallel with $\ell_1 = \ell = r_{258}r_{369}r_{258}r_{147}$. These act on the root variables as their q-shifts:

$$\begin{aligned} &\ell_1(\boldsymbol{a}) = (a_0, a_1, q^{-1}a_2, q^2a_3, q^{-1}a_4, a_5, q^{-1}a_6),\\ &\ell_2(\boldsymbol{a}) = (q^{-1}a_0, q^{-1}a_1, qa_2, q^{-1}a_3, qa_4, q^{-1}a_5, qa_6),\\ &\ell_3(\boldsymbol{a}) = (qa_0, qa_1, a_2, q^{-1}a_3, a_4, qa_5, a_6). \end{aligned} \tag{4.1}$$

Note that ℓ_i's are mutually commutable and $\ell_1\ell_2\ell_3 = \mathrm{id}$. The action of ℓ_i on the auxiliary variables

$$a = (a_0a_1a_5)^{1/3}, \quad b = (a_2a_4a_6q)^{1/3}, \tag{4.2}$$

is described as follows:

$$\ell_1(a, b) = (a, q^{-1}b), \quad \ell_2(a, b) = (q^{-1}a, qb), \quad \ell_3(a, b) = (qa, b). \tag{4.3}$$

Lemma 4.1. *It holds that*

$$\tau_3\ell_3(\tau_6) - a^2b\ell_3(\tau_3)\tau_6 = \left(\frac{{a_1}^2a_2}{{a_0}^2a_6}\right)^{1/3} \frac{1 - a^6b^3}{a^2b}\tau_7\tau_8. \tag{4.4}$$

Proof. We have (see Section 3)

$$\begin{aligned} F_{h-e_3-e_9}(x, y, z) &= a_0a_3{a_4}^2{a_5}^2a_6x + \frac{a_0a_6y}{a_4a_5} + \frac{z}{{a_0}^2a_3a_4a_5{a_6}^2},\\ F_{h-e_6-e_9}(x, y, z) &= \frac{a_4a_5x}{a_1a_2} + \frac{y}{a_1a_2a_3{a_4}^2{a_5}^2} + {a_1}^2{a_2}^2a_3a_4a_5z. \end{aligned}$$

Eliminating x and y, we get

$$F_{h-e_3-e_9} - a_0a_1a_2a_3a_4a_5a_6F_{h-e_6-e_9} = \frac{1 - (a_0a_1a_2a_3a_4a_5a_6)^3}{{a_0}^2a_3a_4a_5{a_6}^2}z. \tag{4.5}$$

Recall that $z = c_z\tau_7\tau_8\tau_9$ and $F_{h-e_i-e_j} = \tau_i\tau_j\tau(h - e_i - e_j)$. By virtue of $\ell_3(e_6) = h - e_3 - e_9$ and $\ell_3(e_3) = h - e_6 - e_9$, we thus obtain (4.4) from (4.5). □

We shall rename the τ-functions as follows:

$$U_{\{1,2,3\}} = \frac{\tau_{\{1,4,7\}}}{N(a, b)}, \quad V_{\{1,2,3\}} = \frac{\tau_{\{2,5,8\}}}{N(q^{1/3}a, q^{-2/3}b)}, \quad W_{\{1,2,3\}} = \frac{\tau_{\{3,6,9\}}}{N(q^{-1/3}a, q^{-1/3}b)}, \tag{4.6}$$

where the normalization factor $N(a, b)$ is defined by

$$N(a, b) = \frac{\left(-\frac{aq}{b}, -ab^2q, -\frac{q}{a^2b}; q, q\right)_\infty \left(\frac{b^3q^3}{a^3}, \frac{q^3}{a^3b^6}, a^6b^3q^3; q^3, q^3\right)_\infty}{\left(\frac{b^2q^2}{a^2}, \frac{q^2}{a^2b^4}, a^4b^2q^2; q^2, q^2\right)_\infty}. \tag{4.7}$$

Equation (4.4) in Lemma 4.1 is then rewritten into

$$\frac{1}{a}W_1\ell_3(W_2) - ab\ell_3(W_1)W_2 = \left(\frac{{a_1}^2a_2}{{a_0}^2a_6}\right)^{1/3} \left(\frac{1}{a} - ab\right)U_3V_3, \tag{4.8}$$

by straightforward computation. As seen below, all the other bilinear equations for U_i, V_i and W_i can also be derived from (4.8) by suitable symmetries of $\widetilde{W}(E_6^{(1)})$. Applying $r_{13}r_{46}r_{79}$ to (4.8) and viewing that $\ell_1 = r_{13}r_{46}r_{79}\ell_3 r_{13}r_{46}r_{79}$, we thus obtain

$$abU_1\ell_1(U_2) - \frac{q}{b}\ell_1(U_1)U_2 = \left(\frac{a_0{a_6}^2}{a_1{a_2}^2}\right)^{1/3}\left(ab - \frac{q}{b}\right)V_3W_3. \tag{4.9}$$

Moreover, we consider an element $\pi = s_0s_1s_5\iota_1\iota_2 \in \widetilde{W}(E_6^{(1)})$ of order six whose action is given as follows:

$$\pi : (a_0, a_1, a_2, a_3, a_4, a_5, a_6; \tau_{\{1,2,3,4,5,6,7,8,9\}}) \mapsto \left(\frac{1}{a_5}, \frac{1}{a_0}, a_0a_6, a_3, a_1a_2, \frac{1}{a_1}, a_4a_5; \tau_{\{7,9,8,1,3,2,4,6,5\}}\right).$$

Hence we see that

$$\pi : (a, b; U_i, V_i, W_i) \mapsto \left(\frac{1}{a}, ab; U_{i-1}, W_{i-1}, V_{i-1}\right), \tag{4.10}$$

and also that the commutation relations $\pi\ell_1 = \ell_1\pi$, $\pi\ell_2 = \ell_3\pi$ and $\pi\ell_3 = \ell_2\pi$ hold. Note that π realizes the rotational diagram automorphism of $A_5^{(1)}$, considered in Remark 3.4. Applying π to (4.8) and (4.9), we get the following proposition.

Proposition 4.2. *The following bilinear equations among the τ-functions U_i, V_i and W_i hold*:

$$abU_i\ell_1(U_{i+1}) - \frac{q}{b}\ell_1(U_i)U_{i+1} = \gamma_i\left(ab - \frac{q}{b}\right)V_{i+2}W_{i+2}, \tag{4.11a}$$

$$\frac{1}{b}V_i\ell_2(V_{i+1}) - \frac{1}{a}\ell_2(V_i)V_{i+1} = \delta_i\left(\frac{1}{b} - \frac{1}{a}\right)W_{i+2}U_{i+2}, \tag{4.11b}$$

$$\frac{1}{a}W_i\ell_3(W_{i+1}) - ab\ell_3(W_i)W_{i+1} = \epsilon_i\left(\frac{1}{a} - ab\right)U_{i+2}V_{i+2}, \tag{4.11c}$$

for $i \in \mathbb{Z}/3\mathbb{Z}$. Here γ_i, δ_i and ϵ_i are the parameters defined by

$$\begin{aligned}
\gamma_1 &= \left(\frac{a_0{a_6}^2}{a_1{a_2}^2}\right)^{1/3}, \quad \gamma_2 = \left(\frac{a_1{a_2}^2}{{a_4}^2a_5}\right)^{1/3}, \quad \gamma_3 = \left(\frac{{a_4}^2a_5}{a_0{a_6}^2}\right)^{1/3},\\
\delta_1 &= \left(\frac{a_0a_2}{a_1a_6}\right)^{1/3}, \quad \delta_2 = \left(\frac{a_1a_4}{a_2a_5}\right)^{1/3}, \quad \delta_3 = \left(\frac{a_5a_6}{a_0a_4}\right)^{1/3},\\
\epsilon_1 &= \left(\frac{{a_1}^2a_2}{{a_0}^2a_6}\right)^{1/3}, \quad \epsilon_2 = \left(\frac{a_4{a_5}^2}{{a_1}^2a_2}\right)^{1/3}, \quad \epsilon_3 = \left(\frac{{a_0}^2a_6}{a_4{a_5}^2}\right)^{1/3}.
\end{aligned} \tag{4.12}$$

We call system (4.11) the *bilinear form of the q-Painlevé equation of type $E_6^{(1)}$*. Conversely, we can verify that, for any functions U_i, V_i, W_i ($i \in \mathbb{Z}/3\mathbb{Z}$) satisfying (4.11), the pair (f, g) defined by

$$f = \frac{U_1V_1W_1}{U_3V_3W_3}, \quad g = \frac{U_2V_2W_2}{U_3V_3W_3},$$

certainly solves the q-Painlevé equation (3.11); here we recall (3.6) and (4.6).

5 Similarity reduction of lattice q-UC hierarchy to q-$P(E_6)$

We shall explain how the bilinear form of q-$P(E_6)$, (4.11), arises naturally from the lattice q-UC hierarchy, through certain periodic and similarity reductions. Let $I = \{1, 2, 3\}$ and $J = \emptyset$ and consider the lattice q-UC hierarchy:

$$t_i T_i(\sigma_{m,n+1}) T_j(\sigma_{m+1,n}) - t_j T_j(\sigma_{m,n+1}) T_i(\sigma_{m+1,n}) = (t_i - t_j) T_{ij}(\sigma_{m,n}) \sigma_{m+1,n+1}. \tag{5.1}$$

We impose the $(3, 3)$-periodic condition:

$$\sigma_{m,n} = \sigma_{m+3,n} = \sigma_{m,n+3}, \tag{5.2}$$

and the similarity condition:

$$\sigma_{m,n}(ct_1, ct_2, ct_3) = c^{d_{m,n}} \sigma_{m,n}(t_1, t_2, t_3), \tag{5.3}$$

for any $c \in \mathbb{C}^\times$. Here $d_{m,n}$ are constant parameters such that $d_{m,n} + d_{m+1,n+1} = d_{m+1,n} + d_{m,n+1}$. We introduce the functions $\widetilde{\sigma}_{m,n}(a, b)$ in two variables defined by $\widetilde{\sigma}_{m,n}(a, b) = \sigma_{m,n}(t_1, t_2, t_3)$ under the substitution $(t_1, t_2, t_3) = (a^{-1}, b^{-1}, ab)$. We thus have the following lemma.

Lemma 5.1. *Let*

$$\begin{aligned}
U_i(a, b) &= \widetilde{\sigma}_{i,-i}(a, b),\\
V_i(a, b) &= \widetilde{\sigma}_{i+1,-i+1}(q^{1/3}a, q^{-2/3}b),\\
W_i(a, b) &= \widetilde{\sigma}_{i+2,-i+2}(q^{-1/3}a, q^{-1/3}b),
\end{aligned} \tag{5.4}$$

for $i \in \mathbb{Z}/3\mathbb{Z}$. Then these functions satisfy the bilinear form of q-$P(E_6)$, (4.11)*, with the parameters*:

$$\gamma_i = q^{(d_{i,-i+2} - d_{i+1,-i})/3}, \quad \delta_i = q^{(d_{i+1,-i} - d_{i+2,-i+1})/3}, \quad \epsilon_i = q^{(d_{i+2,-i+1} - d_{i,-i+2})/3}. \tag{5.5}$$

Proof. Being attentive to the action of ℓ_i's on variables a and b (see (4.3)), one can deduce the bilinear form of q-$P(E_6)$ straightforwardly from the lattice q-UC hierarchy (5.1) by the similarity condition (5.3) together with the periodicity (5.2).

For instance, we shall start from (5.1) with $(m, n) = (r + 1, -r)$ and $(i, j) = (1, 2)$:

$$\begin{aligned}
&t_1 \sigma_{r+1,-r+1}(qt_1, t_2, t_3) \sigma_{r+2,-r}(t_1, qt_2, t_3) - t_2 \sigma_{r+1,-r+1}(t_1, qt_2, t_3) \sigma_{r+2,-r}(qt_1, t_2, t_3)\\
&= (t_1 - t_2) \sigma_{r+1,-r}(qt_1, qt_2, t_3) \sigma_{r+2,-r+1}(t_1, t_2, t_3).
\end{aligned}$$

By using the homogeneity (5.3), we have

$$\begin{aligned}
&q^{(d_{r+1,-r+1} + d_{r+2,-r})/3} t_1 \sigma_{r+1,-r+1}(q^{2/3}t_1, q^{-1/3}t_2, q^{-1/3}t_3) \sigma_{r+2,-r}(q^{-1/3}t_1, q^{2/3}t_2, q^{-1/3}t_3)\\
&- q^{(d_{r+1,-r+1} + d_{r+2,-r})/3} t_2 \sigma_{r+1,-r+1}(q^{-1/3}t_1, q^{2/3}t_2, q^{-1/3}t_3) \sigma_{r+2,-r}(q^{2/3}t_1, q^{-1/3}t_2, q^{-1/3}t_3)\\
&= q^{2d_{r+1,-r}/3} (t_1 - t_2) \sigma_{r+1,-r}(q^{1/3}t_1, q^{1/3}t_2, q^{-2/3}t_3) \sigma_{r+2,-r+1}(t_1, t_2, t_3).
\end{aligned}$$

Putting $(t_1, t_2, t_3) = (a^{-1}, b^{-1}, ab)$, therefore we obtain

$$\begin{aligned}
&\frac{1}{a} \widetilde{\sigma}_{r+1,-r+1}(q^{-2/3}a, q^{1/3}b) \widetilde{\sigma}_{r+2,-r}(q^{1/3}a, q^{-2/3}b)\\
&- \frac{1}{b} \widetilde{\sigma}_{r+1,-r+1}(q^{1/3}a, q^{-2/3}b) \widetilde{\sigma}_{r+2,-r}(q^{-2/3}a, q^{1/3}b)\\
&= q^{(d_{r+1,-r} - d_{r+2,-r+1})/3} \left(\frac{1}{a} - \frac{1}{b} \right) \widetilde{\sigma}_{r+1,-r}(q^{-1/3}a, q^{-1/3}b) \widetilde{\sigma}_{r+2,-r+1}(a, b),
\end{aligned}$$

which turns out to coincide with (4.11b) in view of the action of ℓ_2. In the same way, we can derive also (4.11a) and (4.11c). The proof is now complete. □

6 Algebraic solutions of q-Painlevé equation in terms of the universal character

As seen in the preceding section, the q-Painlevé equation of type $E_6^{(1)}$ is in fact equivalent to a similarity reduction of the (periodic) lattice q-UC hierarchy. On the other hand, we have already known that the lattice q-UC hierarchy admits the universal characters as its homogeneous solutions; see Proposition 2.3. Consequently, we obtain in particular a class of algebraic solutions of the q-Painlevé equation in terms of the universal character.

In order to state our result precisely, we first recall the notion of N-core partitions; see, *e.g.*, [10]. A subset $M \subset \mathbb{Z}$ is said to be a *Maya diagram* if $m \in M$ ($m \ll 0$) and $m \notin M$ ($m \gg 0$). Each Maya diagram $M = \{\ldots, m_3, m_2, m_1\}$ corresponds to a unique partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $m_i - m_{i+1} = \lambda_i - \lambda_{i+1} + 1$. For a sequence of integers $\boldsymbol{n} = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N$, let us consider the Maya diagram

$$M(\boldsymbol{n}) = (N\mathbb{Z}_{<n_1} + 1) \cup (N\mathbb{Z}_{<n_2} + 2) \cup \cdots \cup (N\mathbb{Z}_{<n_N} + N),$$

and denote by $\lambda(\boldsymbol{n})$ the corresponding partition. Note that $\lambda(\boldsymbol{n}) = \lambda(\boldsymbol{n} + \boldsymbol{1})$ where $\boldsymbol{1} = (1, 1, \ldots, 1)$. We call a partition of the form $\lambda(\boldsymbol{n})$ an *N-core partition*. It is well-known that a partition λ is N-core if and only if λ has no hook with length of a multiple of N. We have a cyclic chain of the universal characters attached to N-core partitions; see [16, Lemma 2.2].

Lemma 6.1. *It holds that*

$$S_{\left[\left(k_i,\, \lambda(\boldsymbol{n}(i-1))\right),\, \mu\right]} = \pm S_{[\lambda(\boldsymbol{n}(i)),\, \mu]}, \tag{6.1}$$

for arbitrary $\boldsymbol{n} = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N$ and partition μ. Here $\boldsymbol{n}(i) = \boldsymbol{n} + (\overbrace{1, \ldots, 1}^{i}, \overbrace{0, \ldots, 0}^{N-i})$ and $k_i = Nn_i - |\boldsymbol{n}|$ with $|\boldsymbol{n}| = n_1 + n_2 + \cdots + n_N$.

Finally, by virtue of Proposition 2.3 and Lemmas 5.1 and 6.1, we are led to the following expression of algebraic solutions by means of the universal character attached to a pair of three-core partitions. Define a rational function $R_{[\lambda,\mu]} = R_{[\lambda,\mu]}(a, b)$ by (recall (2.1) or (2.9))

$$R_{[\lambda,\mu]}(a, b) = S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y}) = s_{[\lambda,\mu]}(\boldsymbol{t}), \tag{6.2}$$

under the substitution:

$$x_n = \frac{a^{-n} + b^{-n} + (ab)^n}{n(1 - q^n)}, \quad y_n = \frac{a^n + b^n + (ab)^{-n}}{n(1 - q^{-n})}, \tag{6.3}$$

or $(t_1, t_2, t_3) = (a^{-1}, b^{-1}, ab)$ with $I = \{1, 2, 3\}$ and $J = \emptyset$.

Theorem 6.2. *For any $\boldsymbol{m} = (m_1, m_2, m_3), \boldsymbol{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$, let*

$$\begin{aligned} U_i(a, b) &= R_{[\lambda(\boldsymbol{m}(i)),\, \lambda(\boldsymbol{n}(-i))]}(a, b), \\ V_i(a, b) &= R_{[\lambda(\boldsymbol{m}(i+1)),\, \lambda(\boldsymbol{n}(-i+1))]}(q^{1/3}a, q^{-2/3}b), \\ W_i(a, b) &= R_{[\lambda(\boldsymbol{m}(i+2)),\, \lambda(\boldsymbol{n}(-i+2))]}(q^{-1/3}a, q^{-1/3}b). \end{aligned} \tag{6.4}$$

(i) *These functions solve the system of bilinear equations* (4.11) *with the parameters*:

$$\gamma_i = q^{n_{-i} - m_{i+1} + \frac{|\boldsymbol{m}| - |\boldsymbol{n}|}{3}}, \quad \delta_i = q^{n_{-i+1} - m_{i+2} + \frac{|\boldsymbol{m}| - |\boldsymbol{n}|}{3}}, \quad \epsilon_i = q^{n_{-i+2} - m_i + \frac{|\boldsymbol{m}| - |\boldsymbol{n}|}{3}}. \tag{6.5}$$

(ii) *Consequently, the pair of functions*

$$f = \frac{U_1 V_1 W_1}{U_3 V_3 W_3}, \quad g = \frac{U_2 V_2 W_2}{U_3 V_3 W_3}, \tag{6.6}$$

gives an algebraic solution of the q-Painlevé equation of type $E_6^{(1)}$, (3.11)*, when*

$$\begin{aligned} &a_1 = aq^{\frac{|\boldsymbol{m}|+|\boldsymbol{n}|}{3}-m_1-n_3}, \quad a_5 = aq^{\frac{|\boldsymbol{m}|+|\boldsymbol{n}|}{3}-m_2-n_2}, \quad a_0 = aq^{\frac{|\boldsymbol{m}|+|\boldsymbol{n}|}{3}-m_3-n_1}, \\ &a_2 = bq^{\frac{|\boldsymbol{m}|+|\boldsymbol{n}|-1}{3}-m_3-n_2}, \quad a_4 = bq^{\frac{|\boldsymbol{m}|+|\boldsymbol{n}|-1}{3}-m_1-n_1}, \quad a_6 = bq^{\frac{|\boldsymbol{m}|+|\boldsymbol{n}|-1}{3}-m_2-n_3}. \end{aligned} \tag{6.7}$$

Example 6.3. Let us consider the function

$$P_{[\lambda,\mu]}(a,b;q) = (ab)^{|\lambda|+|\mu|} q^{-|\nu|} \prod_{(i,j)\in\lambda} \left(1 - q^{h(i,j)}\right) \prod_{(k,l)\in\mu} \left(q^{h(k,l)} - 1\right) R_{[\lambda,\mu]}(a,b),$$

associated with the algebraic solutions given in Theorem 6.2. Here we denote by $h(i,j)$ the *hook-length*, that is, $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ (see [7]) and let $\nu = (\nu_1, \nu_2, \ldots)$ be a sequence of integers defined by $\nu_i = \max\{0, \mu'_i - \lambda_i\}$. It is interesting that $P_{[\lambda,\mu]}(a,b;q)$ forms a polynomial whose coefficients are all positive integers. A few examples of the *special polynomials* are given below:

λ	μ	$P_{[\lambda,\mu]}(a,b;q)$
$\emptyset$	$\emptyset$	1
(1)	$\emptyset$	$a + b + a^2b^2$
(2)	$\emptyset$	$a^2 + b^2 + a^4b^4 + (1+q)ab(1 + a^2b + ab^2)$
$(1,1)$	$\emptyset$	$q(a^2 + b^2 + a^4b^4) + (1+q)ab(1 + a^2b + ab^2)$
$\emptyset$	(1)	$1 + a^2b + ab^2$
$\emptyset$	(2)	$q(1 + a^4b^2 + a^2b^4) + (1+q)ab(a + b + a^2b^2)$
(1)	(1)	$(1 + q + q^2)a^2b^2 + qab(a^2 + b^2) + q(a+b)(1 + a^3b^3)$
(1)	(2)	$(1 + q + 2q^2 + q^3)a^2b^2(1 + a^2b + ab^2) + q(1+q)ab(a^2 + b^2 + a^4b^4)$ $+q^2(a + b + a^2b^2(a^3 + b^3) + a^4b^4(a^2 + b^2))$

This polynomial is thought of an analogue of the Umemura polynomials which arise from algebraic solutions of the Painlevé differential equations; *cf.* [11].

7 Verification of Proposition 2.3

Take an $(l + l' + 2) \times (l + l' + 2)$ matrix of the form:

$$\begin{aligned} X &= (X_{a,b})_{1\le a,b\le l+l'+2} \\ &= \begin{array}{c} \left(\begin{array}{c|c|c} -t_i{}^{-1} T_j(H_{\mu_{l'-a+1}+a-1}) & -t_j{}^{-1} T_i(H_{\mu_{l'-a+1}+a-1}) & T_{ij}(H_{\mu_{l'-a+1}+a-b+2}) \\ \hline T_j(h_{\lambda_{a-l'-2}-a+2}) & T_i(h_{\lambda_{a-l'-2}-a+2}) & T_{ij}(h_{\lambda_{a-l'-2}-a+b}) \end{array}\right) \\ \underbrace{\hphantom{-t_i{}^{-1} T_j(H_{\mu_{l'-a+1}+a-1})}}_{1}\ \underbrace{\hphantom{-t_j{}^{-1} T_i(H_{\mu_{l'-a+1}+a-1})}}_{1}\ \underbrace{\hphantom{T_{ij}(H_{\mu_{l'-a+1}+a-b+2})}}_{l+l'} \end{array} \begin{array}{l} \} \, l'+1 \\ \} \, l+1 \end{array} . \end{aligned} \tag{7.1}$$

Let $D = \det X$ and denote by $D[i_1, i_2, \ldots; j_1, j_2, \ldots]$ its minor determinant removing rows $\{i_a\}$ and columns $\{j_a\}$. We put $\lambda_0 = k$ and $\mu_0 = k'$.

Lemma 7.1. *It holds that*

$$(t_i - t_j)s_{[(k,\lambda),(k',\mu)]}(\boldsymbol{t}) = (t_it_j)^{l'+1}D, \tag{7.2a}$$
$$T_{ij}(s_{[\lambda,\mu]}(\boldsymbol{t})) = D[l'+1, l'+2; 1, 2], \tag{7.2b}$$
$$T_i(s_{[(k,\lambda),\mu]}(\boldsymbol{t})) = (-t_j)^{l'}D[l'+1; 1], \tag{7.2c}$$
$$T_j(s_{[\lambda,(k',\mu)]}(\boldsymbol{t})) = (-t_i)^{l'+1}D[l'+2; 2], \tag{7.2d}$$
$$T_j(s_{[(k,\lambda),\mu]}(\boldsymbol{t})) = (-t_i)^{l'}D[l'+1; 2], \tag{7.2e}$$
$$T_i(s_{[\lambda,(k',\mu)]}(\boldsymbol{t})) = (-t_j)^{l'+1}D[l'+2; 1]. \tag{7.2f}$$

Proof. Let us prove only (7.2a) in the following; the others (7.2b)–(7.2f) can be verified in a similar manner. It is easy to see that

$$T_i(h_n) = h_n - t_ih_{n-1}, \tag{7.3a}$$
$$T_i(H_n) = H_n - t_i^{-1}H_{n-1}. \tag{7.3b}$$

We shall apply elementary transformations successively to the row vector $(h_n, h_{n+1}, \dots, h_{n+r-1})$ of size $r = l + l' + 2$. First we add the b^{th} column multiplied by $-t_i$ to the $(b+1)^{\text{th}}$ column for $1 \le b \le r-1$. We then obtain by (7.3a),

$$(h_n, T_i(h_{n+1}), T_i(h_{n+2}), \dots, T_i(h_{n+r-1}))\,.$$

Secondly adding the b^{th} column multiplied by $-t_j$ to the $(b+1)^{\text{th}}$ column for $2 \le b \le r-1$, we get

$$\Big(h_n, T_i(h_{n+1}), T_{ij}(h_{n+2}), \dots, T_{ij}(h_{n+r-1})\Big)\,.$$

Adding the second column multiplied by $(t_i - t_j)^{-1}$ to the first column, we finally obtain the vector:

$$\Big((t_i - t_j)^{-1}T_j(h_{n+1}), T_i(h_{n+1}), T_{ij}(h_{n+2}), \dots, T_{ij}(h_{n+r-1})\Big)\,.$$

By the same procedure as above, the low vector $(H_n, H_{n-1}, \dots, H_{n-r+1})$ is also converted to

$$\Big(-(t_i - t_j)^{-1}t_jT_j(H_n), -t_iT_i(H_n), t_it_jT_{ij}(H_n), \dots, t_it_jT_{ij}(H_{n-r+3})\Big)\,,$$

via (7.3b).

Therefore, remembering (2.9), we arrive at the expression (7.2a). □

Proof of Proposition 2.3. By the use of Jacobi's identity:

$$DD[l'+1, l'+2; 1, 2] = D[l'+1; 1]D[l'+2; 2] - D[l'+1; 2]D[l'+2; 1],$$

we see that (2.7) follows immediately from Lemma 7.1. □

A Reductions to q-Painlevé equations of types $A^{(1)}_{2g+1}$ and $D^{(1)}_5$

Recall that the q-Painlevé equations of types $A^{(1)}_{2g+1}$ and $D^{(1)}_5$ can be derived as reductions from the q-UC hierarchy; see [17] and [18]. Accordingly, they can be derived also from the lattice q-UC hierarchy, as the latter hierarchy includes the former one; see Remark 2.5. We verify that the equations of types $A^{(1)}_{2g+1}$ and $D^{(1)}_5$ are in fact similarity reductions of the lattice q-UC hierarchy together with periodic conditions of order $(g+1, g+1)$ and $(2,2)$, respectively. In this appendix, we demonstrate how to obtain the q-Painlevé equation only for the case of type $D^{(1)}_5$; the other case is simpler, so it may be left to the reader; *cf.* [17].

Let $I = \{1,2\}$ and $J = \{-1,-2\}$. Suppose that $\sigma_{m,n} = \sigma_{m,n}(\boldsymbol{t})$ is a solution of the lattice q-UC hierarchy (2.4), satisfying the periodic condition $\sigma_{m,n} = \sigma_{m+2,n} = \sigma_{m,n+2}$ and the similarity condition $\sigma_{m,n}(c\boldsymbol{t}) = c^{d_{m,n}}\sigma_{m,n}(\boldsymbol{t})$. Here $d_{m,n}$ are constants balanced as $d_{m,n} + d_{m+1,n+1} = d_{m+1,n} + d_{m,n+1}$. Now let us introduce the function $\rho_{m,n}(\alpha,\beta;x)$ in x, equipped with constant parameters α and β, defined by $\rho_{m,n}(\alpha,\beta;x) = \sigma_{m,n}(\boldsymbol{t})$ under the substitution $\boldsymbol{t} = (t_1,t_2,t_{-1},t_{-2}) = (\alpha, \alpha^{-1}, -q^{-1}\beta x, -q^{-1}\beta^{-1}x)$. Let

$$\begin{aligned}
&\Phi^{(-)}_i(x) = \rho_{i,i}(\alpha,\beta;x), && \Phi^{(+)}_i(x) = \rho_{i,i}(q^{1/2}\alpha, q^{1/2}\beta; x),\\
&\Psi^{(-)}_i(x) = \rho_{i,i+1}(\alpha, q^{1/2}\beta; q^{1/2}x), && \Psi^{(+)}_i(x) = \rho_{i,i+1}(q^{1/2}\alpha, \beta; q^{1/2}x),
\end{aligned} \tag{A.1}$$

for $i \in \mathbb{Z}/2\mathbb{Z}$. As similar to the case of $E^{(1)}_6$ (see Section 5), we therefore obtain, from (2.4) with the above constraints, the following system of bilinear equations:

$$\begin{aligned}
&\alpha^{\pm1}q^{(d_{i,i}-d_{i,i+1})/2}\Phi^{(\pm)}_i(x)\Phi^{(\mp)}_{i+1}(x) + \beta^{\pm1}xq^{(d_{i+1,i}-d_{i,i})/2}\Phi^{(\mp)}_i(x)\Phi^{(\pm)}_{i+1}(x)\\
&= \left(\alpha^{\pm1} + \beta^{\pm1}x\right)\Psi^{(\pm)}_i(q^{-1}x)\Psi^{(\mp)}_{i+1}(x),
\end{aligned} \tag{A.2a}$$

$$\begin{aligned}
&\alpha^{\pm1}q^{(d_{i,i+1}-d_{i,i})/2}\Psi^{(\pm)}_i(x)\Psi^{(\mp)}_{i+1}(x) + \left(q^{1/2}\beta\right)^{\mp1}\left(q^{1/2}x\right)q^{(d_{i+1,i}-d_{i,i})/2}\Psi^{(\mp)}_i(x)\Psi^{(\pm)}_{i+1}(x)\\
&= \left(\alpha^{\pm1} + (q^{1/2}\beta)^{\mp1}q^{1/2}x\right)\Phi^{(\pm)}_i(x)\Phi^{(\mp)}_{i+1}(qx),
\end{aligned} \tag{A.2b}$$

where $i \in \mathbb{Z}/2\mathbb{Z}$. We take the variables

$$f(x) = \frac{\Phi^{(+)}_1(x)\Phi^{(-)}_2(x)}{\Phi^{(-)}_1(x)\Phi^{(+)}_2(x)}, \quad g(x) = \frac{\Psi^{(+)}_1(x)\Psi^{(-)}_2(x)}{\Psi^{(-)}_1(x)\Psi^{(+)}_2(x)}, \tag{A.3}$$

and let $\gamma = q^{(d_{1,1}-d_{1,2})/2}$ and $\delta = q^{(d_{2,1}-d_{1,1})/2}$. Hence it follows from (A.2) that

$$\overline{f}f = \frac{(g + \alpha^{-1}\beta^{-1}\gamma\delta x)(g + \alpha\beta\gamma^{-1}\delta^{-1}qx)}{(xg + \alpha\beta\gamma\delta)(qxg + \alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1})}, \tag{A.4a}$$

$$\underline{g}g = \frac{(f + \alpha^{-1}\beta\gamma^{-1}\delta x)(f + \alpha\beta^{-1}\gamma\delta^{-1}x)}{(xf + \alpha\beta^{-1}\gamma^{-1}\delta)(xf + \alpha^{-1}\beta\gamma\delta^{-1})}, \tag{A.4b}$$

where the symbols $\overline{f}$ and $\underline{g}$ stand for $f(qx)$ and $g(q^{-1}x)$, respectively. This system is equivalent to the q-Painlevé equation of type $D^{(1)}_5$, known as the q-Painlevé VI equation; see [1].

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